Power Spectrum of Hard-Limited Gaussian Processes

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The power spectral density at the output of an ideal hard limiter (one-bit quantizer) is examined when the input is driven by a narrowband gaussian signal plus an additive gaussian noise that consists of a broadband background component plus narrowband interference. Assuming that the input signal-to-noise power ratio is small by virtue of the large bandwidth of the observed broadband noise, calculations are made of the average output signal power, the average output noise power in the signal band, and the average power of the strongest intermodulation product. The results support the intuitive conclusion that spectrum analyzer performance is degraded by the presence of the limiter and that this degradation is more pronounced when a strong narrowband interfering signal is present. They also indicate that the degradation can be minimized by making the bandwidth observed by the limiter sufficiently wide that the broadband noise power dominates both the signal and interference powers. In particular, for a typical example, the signal-to-noise power ratio measured in the signal band is degraded by less than about 1.3 dB by the presence of the limiter and the ratio of output signal power to power of the strongest intermodulation product is greater than about 14.5 dB as long as the broadband noise power exceeds the interfering-signal power.

I. INTRODUCTION

In this paper we examine the power spectral density at the output of an ideal hard limiter when the input is driven by a collection of independent gaussian processes. This work is motivated hy the fact that in spectrum analysis, it is often convenient from the point of view of signal processing to precede the analyzer with a hard limiter. In order to determine the effect of the limiter on analyzer performance, it is of interest to compare the power spectral density at the limiter output with that at the limiter input. With this goal in mind, the ideal limiter to he ana-

lyzed is shown in Fig. 1. It is assumed that the limiter input is driven by the signal

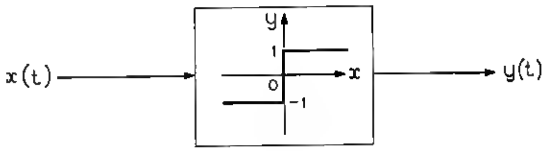

Fig. 1 — Ideal hard limiter.

$$x(t) = s(t) + n(t), \tag{1}$$

where $s(t)$ is a sample function of the gaussian "signal" process $S(t)$ and $n(t)$ is a sample function of the gaussian "noise" process $N(t)$. More precisely, it is assumed that $S(t)$ and $N(t)$ are statistically independent, zero-mean, stationary, real, gaussian processes having continuous covariance functions $R_S(\tau)$ and $R_N(\tau)$ respectively. Further, motivated by the spectrum analysis application, the covariance functions $R_S(\tau)$ and $R_N(\tau)$ are specified: the signal process $S(t)$ is assumed to be a narrowband process with covariance function

$$R_S(\tau) = R_0(\tau) \cos \omega_0 \tau \tag{2}$$

where $S_0(f)$, the Fourier transform of $R_0(\tau)$, occupies a narrow band centered at zero frequency. The noise process $N(t)$ is assumed to consist of a broadband background component plus narrowband interference that is statistically independent of the background noise. The covariance function of the broadband background noise is assumed to be a continous covariance function that is given in the form†

$$\begin{aligned} R_1(\tau) &= R_1(\tau_1\ ;\ \tau) \\ &= \frac{C_0}{\tau_1}\, \rho\left(\frac{|\tau|}{\tau_1}\right) \cos \omega_1 \tau, \end{aligned} \tag{3}$$

where $\rho(x)$ satisfies the conditions

$$\rho(0) = 1, \tag{4}$$

$$\int_0^\infty |\rho(x)|\, dx < \infty. \tag{5}$$

This specification of $R_1(\tau)$ has the properties:

† For example, consider the exponential covariance

$$R_{1E}(\tau) = \frac{C_0}{\tau_1} \exp\left(-\beta \frac{|\tau|}{\tau_1}\right) \cos \omega_1 \tau.$$

(*i*) The total average broadband noise power $R_1(0)$ increases linearly with τ_1^{-1} where $\tau_1 > 0$ is defined to be the broadband noise "correlation time."

(*ii*) The average broadband noise power observed in any fixed band of finite extent approaches a finite constant as the correlation time τ_1 approaches zero.

Finally, the covariance function of the narrowband interference is assumed to be given by $R_2(\tau) \cos \omega_2\tau$ where $S_2(f)$, the Fourier transform of $R_2(\tau)$, occupies a narrow band centered at zero frequency. Therefore, the covariance function of the noise process $N(t)$ is given by

$$R_N(\tau) = R_1(\tau) + R_2(\tau) \cos \omega_2\tau \tag{6}$$

where $R_1(\tau)$ satisfies equation (3).

It was stated that the covariance functions just specified are suggested by the spectrum analysis application, and this is true in the following sense: it is often the case that one desires to analyze narrowband signals that lie at *a priori* unknown locations within a relatively wide band, and in fact it may be that the total bandwidth to be searched is a significant fraction of the band center frequency. Given such a spectrum analysis problem, it is proposed that the situation of greatest interest is that in which the average noise power in the narrow band actually occupied by the signal may or may not be comparable to the average signal power, but in which the total average noise power is much larger than the average signal power by virtue of the large noise bandwidth. Having such a situation in mind, it is seen that the model for the broadband covariance function $R_1(\tau)$ specified in equation (3) does in fact exhibit the desired behavior when the correlation time τ_1 is appropriately small.

However, in addition to this "weak-signal" situation in which the narrowband signal power $R_S(0)$ is much smaller than the broadband noise power $R_1(0)$, it is also of interest to allow the presence of "strong" narrowband signals whose average power is comparable to that of the broadband background noise. The presence of such strong narrowband signals is expected to be obvious at the limiter output, and in fact these signals are of interest since we expect that their presence will lead to the generation of intermodulation products that may interfere with the analysis of any weak signals that are present. In order to examine this situation, a narrowband interference has been included, and it is convenient to consider this interfering signal to be part of the additive noise $N(t)$.

Before proceeding with the analysis of the problem stated above, it is noted that the ideal limiter described in Fig. 1 has received a great deal of attention in the literature. The noiseless case has been considered

and output amplitudes examined when the input consists of a eollection of sinusoids.[1,2] The noise-alone case has been examined and results obtained for the autocorrelation function and power spectral density at the limiter output both for the case of broadband gaussian noise alone $[R_1(\tau)]$ and for the case of narrowhand gaussian noise alone $[R_2(\tau) \cos \omega_2\tau]$.[3,4] The ratio of output signal-to-noise ratio (SNR) to input SNR has been evaluated for the case in which the input consists of one or two sinusoids plus narrowband gaussian noise.[5-7] These same workers have examined the strengths of intermodulation products, and the analysis of output signal and intermodulation product power has been extended to the case of an arhitrary numher of sinusoids plus gaussian noise.[8,9] In addition, analysis of the limiter has played an important part in studies of tbe performance of angle-modulation systems, and these analyses have generally assumed tbat the limiter is driven by a narrowband process.

On the other hand, it does not appear that much has been reported for tbe situation in which the limiter is driven hy a narrowhand signal plus noise that includes a hroadhand component. Known results that have applieation to this situation include those of Manasse, and others, which apply when the limiter is driven by a "weak" narrowband signal plus narrowband gaussian noise whose bandwidth is much larger than that of tbe signal,[10] plus approximate results that apply when the input includes a narrowband component that is "much strongcr" than the sum of the other inputs present.[11] We address this problem by examining the the output power spectral density when the limiter input is given hy cquation (1); namely, the input is made up of a narrowband gaussian signal plus a gaussian noise consisting of a hroadband background component plus narrowhand interference. In particular, this examination is carried out by calculating the output power spectral density in Section II, as the hroadband noise correlation time τ_1 approaches zero. This calculated result is then used in Section III to evaluate three performance measures. An example of a system to which these performance measures apply is a spectrum analyzer preceded hy the ideal limiter.

(*i*) The degradation in the ratio (SNR) of average signal power to average noise power in the spectral hand occupied by the signal is calculated. This degradation is important bccause the signal-to-noise power ratio measured in the signal hand is often one of the important parameters in determining system performance.

(*ii*) The ratio (SIR) of average output signal power to average image

power is calculated where, if the narrowband signal is centered at a frequency f_0 and the narrowband interference is centered at a frequency f_2, then the signal image is defined to be that intermodulation product centered at the frequency $| 2f_2 - f_0 |$. This is the strongest of the intermodulation products of the signal with the additive noise, and thus it is reasonable to use the SIR as an indication of whether or not these intermodulation products will have a significant effect on system performance.

(*iii*) The ratio S_2NR_0 of average output interference power to average output broadband noise power in the spectral band occupied by the interference is calculated. As discussed previously, the distinction in this work between signal and interference is made based upon average power at the limiter input. That is, it has been assumed that the presence of any narrowband signal having an average power comparable to that of the broadband background noise will be obvious at the limiter output, and that such an input may in fact interfere with the analysis of other narrowband inputs. S_2NR_0 is calculated to check the assumption that in fact the presence and location of such an interfering signal will be obvious upon analyzing the power spectrum at the limiter output.

Since the performance measures listed above are calculated as the broadband noise correlation time τ_1 approaches zero, it follows that they will all apply in practice to situations in which the broadband component of the input noise has been shaped by a low-pass filter whose bandwidth is large compared with the center frequencies of the narrowband inputs that may be present. An example of a situation in which such a model is viable occurs in the spectrum analysis of underwater acoustical signals.

On the other hand, the SNR and S_2NR_0 results obtained will not apply directly to communication situations in which the bandwidth of the additive broadband noise is much larger than that of the narrowband signal but much smaller than the system center frequency. This situation is discussed in Section IV, and it is pointed out there that the results can be modified to encompass this situation by letting the center frequencies of both the narrowband signal and additive noise increase linearly with τ_1^{-1}.

II. THE OUTPUT POWER SPECTRAL DENSITY

The output power spectral density can be calculated by using the expression for the output autocorrelation function $R_Y(\tau)$ given by Davenport and Root (Ref. 12, p. 308)

$$R_Y(\tau) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{2^{k+m}\Gamma^2[(k+m)/2]}{\pi^2 k!\, m!\, [R_S(0) + R_N(0)]^{k+m}} R_S^k(\tau) R_N^m(\tau), \quad k + m \text{ odd}$$

$$= 0, \qquad \text{otherwise,} \tag{7}$$

where $\Gamma(x)$ denotes the gamma function; in conjunction with the expression for $R_Y(\tau)$ given by Van Vleck (Ref. 3, p. 23)

$$R_Y(\tau) = \frac{2}{\pi} \arcsin \left[\frac{R_S(\tau) + R_N(\tau)}{R_S(0) + R_N(0)} \right]. \tag{8}$$

Defining α to be the fraction of the average noise power due to the broadband background noise,

$$\alpha \triangleq \frac{R_1(0)}{R_N(0)} = \frac{R_1(0)}{R_1(0) + R_2(0)}, \tag{9}$$

it is seen that the ratio η_S of average signal power to average noise power at the limiter input is given by

$$\eta_S \triangleq \frac{R_S(0)}{R_N(0)} = \alpha \frac{R_S(0)}{C_0} \tau_1 . \tag{10}$$

Now, it was pointed out in Section I that we are interested in the situation in which the signal-to-noise power ratio η_S is small, and in fact the case of interest is that in which η_S is small because τ_1 is small, that is, η_S is small due to the large bandwidth of the observed broadband background noise. Motivated by this, it is shown in Appendix A, using the expressions for $R_Y(\tau)$ given by equations (7) and (8), that when $\alpha > 0$ the output power spectral density $S_Y(f)$ is given by

$$S_Y(f) = \frac{4}{\pi} \left\{ \int_0^\infty \arcsin \rho_N(\tau) \cos \omega\tau \, d\tau \right.$$
$$+ \alpha \frac{R_S(0)}{C_0} \tau_1 \int_0^\infty [\rho_S(\tau) - (1-\alpha)\rho_2(\tau) \cos \omega_2\tau]$$
$$\left. \cdot [1 - (1-\alpha)^2 \rho_2^2(\tau) \cos^2 \omega_2\tau]^{-\frac{1}{2}} \cos \omega\tau \, d\tau \right\} + o(\tau_1) \tag{11}$$

as $\tau_1 \to 0$, uniformly in f, where

$$\rho_\gamma(\tau) \triangleq \frac{R_\gamma(\tau)}{R_\gamma(0)}, \qquad \gamma = S, N, 0, 1, 2, \tag{12}$$

are assumed to be absolutely integrable.

Equation (11) exhibits the components that dominate the output power spectral density when the broadband noise correlation time τ_1

approaches zero. In particular, inspection of equation (11) shows that these dominant contributions include a component that is just the output power spectral density observed when the noise $N(t)$ alone is present at the limiter input, a component tbat has the spectral characteristics of the signal $S(t)$, and a component that is due to interaction of the signal with the interference component $[\rho_2(\tau) \cos \omega_2\tau]$ of the noise. In order to quantitatively analyze these components where, in particular, we desire to use $S_Y(f)$ to calculate the performance measures discussed in Section I it is convenient to make use of the fact that both the signal $S(t)$ and the interference component of the noise have been assumed to be narrowband processes, plus the fact that the hroadhand component of the noise hecomes white across any fixed hand of finite extent when $\tau_1 \to 0$. These properties can be exploited hy expanding both $[1 - (1 - \alpha)^2\rho_2^2(\tau) \cos \omega_2^2\tau]^{-\frac{1}{2}}$ and arcsin $\rho_N(\tau)$ followed hy an appropriate collection of terms. This is carried out in Appendix B and the result is

$$\begin{aligned} S_Y(f) = S_{Y_1}(f) + S_{Y_2}(f) + \frac{4}{\pi^2}\alpha \frac{R_S(0)}{C_0} \tau_1 \sum_{m=0}^{\infty} \epsilon_m \frac{\Gamma^2(m + \frac{1}{2})}{\Gamma(2m + 1)} (1 - \alpha)^{2m} \\ \cdot \int_0^\infty {}_2F_1[m + \tfrac{1}{2}, m + \tfrac{1}{2}; 2m + 1; (1 - \alpha)^2\rho_2^2(\tau)] \\ \cdot \rho_S(\tau)\rho_2^{2m}(\tau) \cos 2m\omega_2\tau \cos \omega\tau \, d\tau \\ - \frac{8}{\pi^2}\alpha \frac{R_S(0)}{C_0} \tau_1 \sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{1}{2})\Gamma(m + \frac{3}{2})}{\Gamma(2m + 2)} (1 - \alpha)^{2m+1} \\ \cdot \int_0^\infty {}_2F_1[m + \tfrac{1}{2}, m + \tfrac{3}{2}; 2m + 2; (1 - \alpha)^2\rho_2^2(\tau)] \\ \cdot \rho_2^{2m+1}(\tau) \cos (2m + 1)\omega_2\tau \cos \omega\tau \, d\tau + o(\tau_1) \end{aligned} \tag{13}$$

as $\tau_1 \to 0$, uniformly in f, where ${}_2F_1(a, b; c; x)$ is Gauss's hypergeometric function (Rcf. 13, p. 556), ϵ_m is the Neumann factor $\epsilon_0 = 1$, $\epsilon_m = 2(m = 1, 2, \cdots)$, and where $S_{Y_1}(f)$ and $S_{Y_2}(f)$ are given:

$$S_{Y_1}(f) = \frac{4}{\pi} \tau_1 \int_0^\infty \{\arcsin [\alpha\rho(x) + 1 - \alpha] - \arcsin (1 - \alpha)\} \, dx + o(\tau_1) \tag{14}$$

as $\tau_1 \to 0$, for all $f \ll f_{\max} < \infty$ for arbitrary fixed $f_{\max}$† and

† Recall from equation (3) that

$$\rho_1(\tau) = \rho\left(\frac{|\tau|}{\tau_1}\right) \cos \omega_1\tau$$

where $\rho(x)$ satisfies equations (4) and (5).

$$S_{Y_3}(f) = \frac{4}{\pi^2} \sum_{m=0}^{\infty} \frac{\Gamma^2(m + \frac{1}{2})}{\Gamma(2m + 2)} (1 - \alpha)^{2m+1}$$
$$\cdot \int_0^{\infty} {}_2F_1[m + \tfrac{1}{2}, m + \tfrac{1}{2}; 2m + 2; (1 - \alpha)^2 \rho_2^2(\tau)]$$
$$\cdot \rho_2^{2m+1}(\tau) \cos (2m + 1)\omega_2 \tau \cos \omega\tau \, d\tau. \tag{15}$$

The expression given by equations (13), (14), and (15) exhibits in a useful fashion the components that dominate the output power spectral density when the broadband noise correlation time approaches zero. To see this more clearly, it is convenient to assume that the narrowband interference in fact has a line spectrum, that is,

$$\rho_2(\tau) \equiv 1. \tag{16}$$

This assumption is convenient since it simplifies the calculations without obscuring the most important effects that result from the presence of narrowband interference. This assumption is applied in Appendix B to equations (13), (14), and (15), and it is shown that, when $\rho_2(\tau) \equiv 1$, we can write

$$\lim_{\tau_1 \to 0} S_Y(f) = S_{Y_1}(f) + S_{Y_3}(f) + \frac{\alpha}{\pi^2 C_0} \tau_1 \sum_{m=0}^{\infty} \epsilon_m \frac{\Gamma^2(m + \frac{1}{2})}{\Gamma(2m + 1)} (1 - \alpha)^{2m}$$
$$\cdot {}_2F_1[m + \tfrac{1}{2}, m + \tfrac{1}{2}; 2m + 1; (1 - \alpha)^2]$$
$$\cdot [S_S(f - 2mf_2) + S_S(f + 2mf_2)] \tag{17}$$

where $S_{Y_1}(f)$ is given by equation (14),

$$S_{Y_3}(f) = \frac{1}{\pi^2} \sum_{m=0}^{\infty} \frac{\Gamma^2(m + \frac{1}{2})}{\Gamma(2m + 2)} (1 - \alpha)^{2m+1}$$
$$\cdot {}_2F_1[m + \tfrac{1}{2}, m + \tfrac{1}{2}; 2m + 2; (1 - \alpha)^2]$$
$$\cdot \{ \delta[f - (2m + 1)f_2] + \delta[f + (2m + 1)f_2] \} \tag{18}$$

where $\delta(x)$ denotes the Dirac-delta function, and where

$$S_S(f) = 2 \int_0^{\infty} R_S(\tau) \cos \omega\tau \, d\tau \tag{19}$$

is the power spectral density of the signal $S(t)$. Equations (17), (14), and (18) give the representation we desire, and they demonstrate that there are three contributions that dominate the output power spectral density when the broadband noise correlation time τ_1 approaches zero.

(*i*) There is a component $S_{Y_1}(f)$ that becomes white across any frequency band of finite extent as $\tau_1 \to 0$. When $\alpha = 1$, this component is

just the output power spectral density that would be observed if the broadband component of the noise was present alone at the limiter input. Moreover it is necessary to specify the broadband covariance function $R_1(\tau)$ in order to calculate $S_{Y_1}(f)$. For example, if $R_1(\tau)$ is the "triangular" covariance function

$$R_{1\Delta}(\tau) \triangleq \frac{C_0}{\tau_1}\left(1 - \frac{|\tau|}{\tau_1}\right), \qquad |\tau| \leqq \tau_1$$
$$\triangleq 0, \qquad |\tau| > \tau_1, \tag{20}$$

then equation (14) gives the result

$$S_{Y_{1\Delta}}(f) = \frac{4}{\pi}\frac{\tau_1}{\alpha}\left[\frac{\pi}{2} - \arcsin(1-\alpha) - (2\alpha - \alpha^2)^{\frac{1}{2}}\right] + o(\tau_1) \tag{21}$$

as $\tau_1 \to 0$, for all $f \leqq f_{\max} < \infty$ for arbitrary fixed $f_{\max}$.

(*ii*) There is a component $S_{Y_2}(f)$ consisting of line spectra located at $|f| = kf_2$, $k = 1, 3, \cdots$. When $\alpha = 0$, this component is just the power spectral density that would be observed if the narrowband interference was present alone at the limiter input.

(*iii*) There is a component consisting of a term that has the spectral characteristics of the signal plus terms that are intermodulation products of the signal with the narrowband interference component of the noise.

2.1 *Noise Consisting of Broadband Component Alone*

It is clear from inspection of equations (17), (14), and (18) that the output power spectral density is greatly simplified when the additive noise consists only of the broadband component ($\alpha = 1$), and in fact it is seen that in this case equation (13) reduces to the simple result

$$S_Y(f) = S_{Y_1}(f) + \frac{2}{\pi}\frac{\tau_1}{C_0} S_S(f) + o(\tau_1) \tag{22}$$

as $\tau_1 \to 0$, uniformly in f. Moreover, the calculation of $S_{Y_1}(f)$ is simplified when $\alpha = 1$. For example, if $R_1(\tau)$ is given by the triangular function in equation (20), then it is seen that, when $\alpha = 1$,

$$S_{Y_{1\Delta}}(f) = \frac{4}{\pi}\int_0^{\tau_1} \arcsin\left(1 - \frac{\tau}{\tau_1}\right)\cos\omega\tau \, d\tau. \tag{23}$$

This integral can be evaluated using Erdelyi [Ref. 14, item 4.8(1)], and we find

$$S_{Y_{1\Delta}}(f) = 2\tau_1\left[J_0(\omega\tau_1)\operatorname{sinc}(2f\tau_1) - \frac{1}{\omega\tau_1}H_0(\omega\tau_1)\cos\omega\tau_1\right] \tag{24}$$

where $J_\nu(x)$ denotes the Bessel function of the first kind of order ν and $H_\nu(x)$ is a Struve function of order ν (Ref. 14, p. 372).† Note that equation (24) holds for all f and for all τ_1 . $S_{Y_{1\Delta}}(f)$ is plotted in Fig. 2 along with

$$S_{1\Delta}(f) = C_0 \operatorname{sinc}^2 (f\tau_1), \tag{25}$$

the power spectral density at the limiter input corresponding to $R_{1\Delta}(\tau)$. The plotted data are normalized so that both processes have the same average power. Thus the data plotted in Fig. 2 show explicitly how the ideal limiter redistributes the average broadband noise power across the band and demonstrate in particular the power-spreading effect that takes place due to the limiter nonlinearity.

III. EVALUATION OF PERFORMANCE MEASURES

It is now desired to use the output power spectral density results derived above to evaluate the performance measures discussed in Section I. These calculations use directly the results derived above except that the assumption that the narrowband interference has a line spectrum can be relaxed. That is, the results derived below continue to be useful as long as the interference is a narrowband gaussian process with the covariance function $R_2(\tau) \cos \omega_2\tau$ specified in Section I.

3.1 *Degradation in Signal-to-Noise Power Ratio*

The degradation in signal-to-noise power ratio in the spectral band occupied by the signal is obtained by calculating the ratio $\mathrm{SNR_0/SNR_I}$ of output signal-to-noise power ratio to input signal-to-noise power ratio, where these SNR's are calculated in the spectral band B occupied by the signal. Moreover, we assume that:

(*i*) The band B contains significant contributions from only the narrowband signal and the broadband component of the noise, that is, the narrowband interference and intermodulation products of the narrowband signal with the narrowband interference have negligible power in the band B.

(*ii*) $R_1(\tau)$ is the triangular function in equation (20) since it is necessary to specify the covariance function of the broadband component of the noise.

Making these assumptions, the ratio $\mathrm{SNR_0/SNR_I}$ measured in the

† Note that $\operatorname{sinc} x \triangleq \dfrac{\sin \pi x}{\pi x}$.

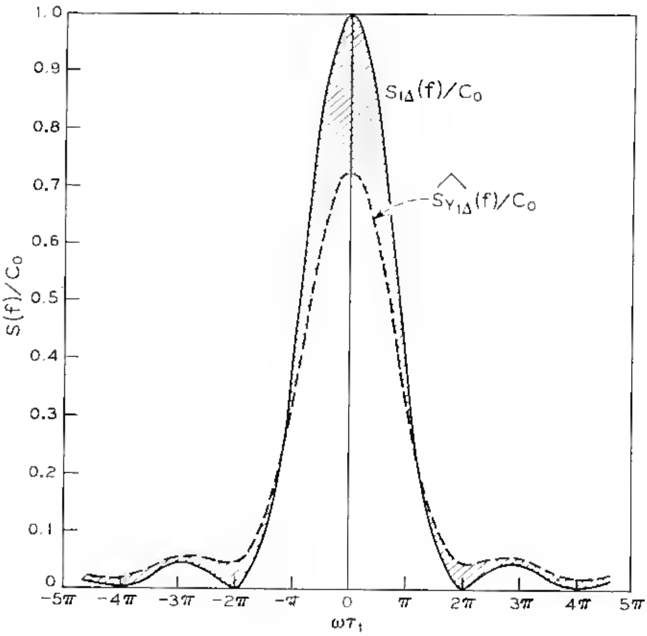

Fig. 2 — Normalized power spectral density.

$$\widehat{S_{Y_1\Delta}(f)} = \frac{R_{1\Delta}(0)}{R_{Y_1\Delta}(0)} S_{Y_1\Delta}(f);\ S_{1\Delta}(f) = C_0 \operatorname{sinc}^2(f\tau_1).$$

band B of finite extent can be calculated using $S_Y(f)$ given by equation (17), and it is seen that

$$\lim_{\tau_1 \to 0} \frac{\mathrm{SNR}_0}{\mathrm{SNR}_1} = \frac{\int_B S_{YS}(f)\,df \int_B S_1(f)\,df}{\int_B S_{Y_1}(f)\,df \int_B S_S(f)\,df} \tag{26}$$

where $S_{Y_1}(f)$ is given by equation (21), $S_S(f)$ is the power spectral density of the narrowband signal $S(t)$, and $S_{YS}(f)$ and $S_1(f)$ are given: $S_{YS}(f)$ is defined to be the contribution to $S_Y(f)$ that has the spectral characteristics of the signal $S(t)$ and thus is determined by setting $m = 0$ in the sum in equation (17). This gives

$$S_{YS}(f) = \frac{2}{\pi}\frac{\alpha}{C_0}\,{}_2F_1[\tfrac{1}{2}, \tfrac{1}{2}; 1; (1-\alpha)^2]\tau_1 S_S(f) \tag{27}$$

which (using p. 387 of Ref. 14) can be written as

$$S_{YS}(f) = \frac{4}{\pi^2}\frac{\alpha}{C_0}K(1-\alpha)\tau_1 S_S(f) \tag{28}$$

where $K(k)$ denotes the complete elliptic integral of the first kind. $S_1(f)$ is defined to he the power spectral density at the limiter input due to the hroadhand component of the noise and thus, using equation (25), is given by

$$S_1(f) = C_0 + O(\tau_1^2) \tag{29}$$

as $\tau_1 \to 0$, for all $f \leqq f_{\max} < \infty$ for arbitrary fixed $f_{\max}$. Thus, making the appropriate substitutions into equation (26) yields

$$\lim_{\tau_1 \to 0} \frac{\mathrm{SNR}_0}{\mathrm{SNR}_I} = \frac{\alpha^2 K(1-\alpha)}{\pi\left[\dfrac{\pi}{2} - \arcsin(1-\alpha) - (2\alpha - \alpha^2)^{\frac{1}{2}}\right]}. \tag{30}$$

This relative signal-to-noise power ratio result is plotted in Fig. 3 and demonstrates the expected result that the degradation in the signal band increases when there is a strong narrowhand interfering signal present at the limiter input. However, it is important to note that the

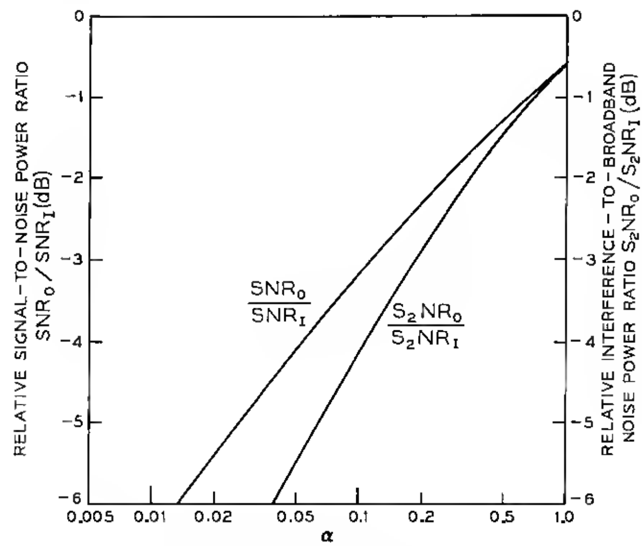

Fig. 3 — Relative signal-to-noise power ratios.

$$\alpha \triangleq \frac{R_1(0)}{R_1(0) + R_2(0)}.$$

narrowband interference must be very strong to cause a significant increase in the degradation. In particular, it is seen that the degradation is less than about 1.3 dB as long as α is greater than 0.5, that is, as long as the broadband noise power is greater than the narrowband interference power.

3.2 *Signal-to-Image Power Ratio*

The signal-to-image power ratio (SIR) is obtained by calculating the ratio of average output signal power to average image power where the image has been defined to be tbat narrowband component of $S_Y(f)$ centered at the frequency $| 2f_2 - f_0 |$. The SIR can be calculated using $S_Y(f)$ given by equation (17), but it should be noted tbat, wben $\tau_1 \to 0$, the SIR does not depend on the particular choice of $R_1(\tau)$ within the class specified by equation (3). Using equation (17), it is seen that

$$\lim_{\tau_1 \to 0} \mathrm{SIR} = \frac{\int_{-\infty}^{\infty} S_{Y_S}(f)\, df}{\frac{1}{2}\int_{-\infty}^{\infty} S_{Y_I}(f)\, df} \tag{31}$$

wbere $S_{Y_S}(f)$ is given by equation (28) and $S_{Y_I}(f)$ is found by setting $m = 1$ in the sum in equation (17). Tbat is,

$$S_{Y_I}(f) = \frac{1}{4\pi}\frac{\alpha(1-\alpha)^2}{C_0}\, {}_2F_1[\tfrac{3}{2}, \tfrac{3}{2}; 3; (1-\alpha)^2]\tau_1 \cdot [S_S(f - 2f_2) + S_S(f + 2f_2)], \tag{32}$$

whicb, using Abramowitz and Stegun [Ref. 13, item 15.2.1] togetber with Price [Ref. 15, p. 10] and Dwight [Ref. 16, items 788.1, 788.2], can be written as

$$S_{Y_I}(f) = \frac{8}{\pi^2}\frac{\alpha}{(1-\alpha)^2 C_0}\left[\frac{1 + 2\alpha - \alpha^2}{2} K(1-\alpha) - E(1-\alpha)\right]\tau_1 \cdot [S_S(f - 2f_2) + S_S(f + 2f_2)] \tag{33}$$

where $E(k)$ denotes the complete elliptic integral of the second kind. Making tbe appropriate substitutions, there results

$$\lim_{\tau_1 \to 0} \mathrm{SIR} = \frac{(1-\alpha)^2 K(1-\alpha)}{(1 + 2\alpha - \alpha^2)K(1-\alpha) - 2E(1-\alpha)}. \tag{34}$$

This SIR result is plotted in Fig. 4 and demonstrates tbat the signal-to-image power ratio decreases when tbere is a strong narrowband interfering signal present at the limiter input. In fact, equation (34)

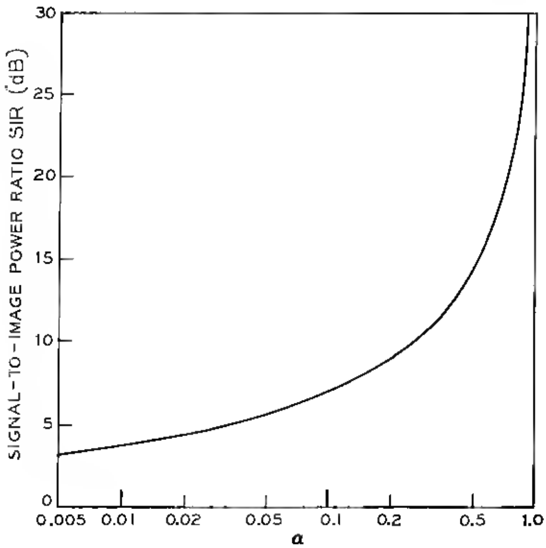

Fig. 4 — Signal-to-image power ratio.

$$\alpha \triangleq \frac{R_1(0)}{R_1(0) + R_2(0)}.$$

has the limiting behavior

$$\lim_{\alpha \to 0} \lim_{r_1 \to 0} \text{SIR} = 1, \tag{35}$$

which agrees with the approximate result obtained when one assumes that the input to the limiter includes a narrowband component tbat is much stronger than the sum of the other input components present.[11] However, tbe most interesting result demonstrated by Fig. 4 is that tbe narrowband interference must be very strong for the image power to be comparable to tbe signal power at the limiter output. In particular, it is seen that the SIR is greater than about 14.5 dB as long as the broadband noise power is greater than the narrowband interference power.

3.3 *Output Interference-to-Broadband Noise Power Ratio*

The output interference-to-broadband noise power ratio S_2NR_0 is obtained by calculating the ratio of average output interference power to average output broadband noise power, measured in the spectral

band B_2† occupied by the interference. In order to perform this calculation it is necessary to specify the broadband covariance function, and is is assumed that $R_1(\tau)$ is the triangular function in equation (20). Having specified $R_1(\tau)$ in this manner, S_2NR_0 can be calculated using $S_Y(f)$ given by equation (17), and it is seen that

$$\lim_{\tau_1 \to 0} S_2NR_0 = \frac{\int_{B_2} S_{Y_2}(f)\, df}{\int_{B_2} S_{Y_1}(f)\, df} \tag{36}$$

where $S_{Y_1}(f)$ is given by equation (21) and $S_{Y_2}(f)$ is given by equation (18). Proceeding with these substitutions and making the assumption that the components of $S_{Y_2}(f)$ concentrated at (odd) harmonics of the fundamental frequency f_2 contribute negligible power in the band B_2, there results

$$\lim_{\tau_1 \to 0} S_2NR_0 = \frac{\alpha(1-\alpha)\,{}_2F_1[\frac{1}{2}, \frac{1}{2}; 2; (1-\alpha)^2]}{2\tau_1\left[\frac{\pi}{2} - \arcsin(1-\alpha) - (2\alpha - \alpha^2)^{\frac{1}{2}}\right]\left(\int_{B_2} df\right)}, \tag{37}$$

which, making use of Price [Ref. 15, p. 10], can be written as

$$\lim_{\tau_1 \to 0} S_2NR_0 = \frac{2\alpha[E(1-\alpha) - (2\alpha - \alpha^2)K(1-\alpha)]}{\pi(1-\alpha)\left[\frac{\pi}{2} - \arcsin(1-\alpha) - (2\alpha - \alpha^2)^{\frac{1}{2}}\right]W\tau_1} \tag{38}$$

where

$$W \triangleq \int_{B_2} df. \tag{39}$$

The normalized power ratio $\lim_{\tau_1 \to 0} W\tau_1(S_2NR_0)$ is plotted in Fig. 5, and the plotted data are seen to support the intuitive assumption made in Section I that the presence and location of a narrowband input having an average power comparable to that of the broadband background noise will be obvious at the limiter output.

A result of perhaps more interest than S_2NR_0 is the ratio S_2NR_0/S_2NR_I of output interference-to-broadband noise power ratio to input interference-to-broadband noise power ratio. This calculation can be carried out in the same way that SNR_0/SNR_I was calculated earlier, and we find

† This calculation is not of interest if the interference truly has a line spectrum (that we can resolve). However, it is of interest here since these results are useful as long as the interference is a narrowband gaussian process.

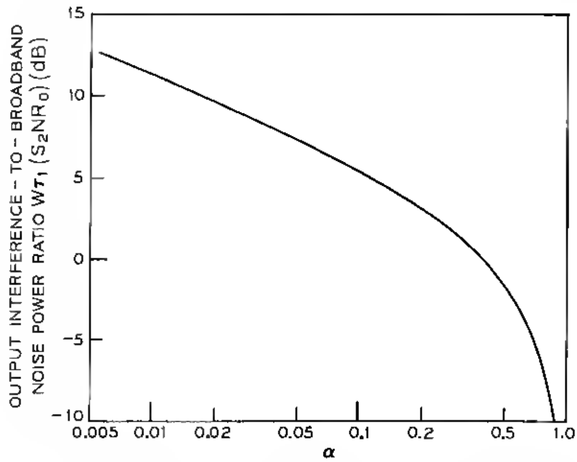

Fig. 5 — Normalized output interference-to-broadband noise power ratio.

$$\alpha = \frac{R_1(0)}{R_1(0) + R_2(0)}.$$

$$\lim_{\tau_1 \to 0} \frac{S_2NR_0}{S_2NR_I} = \frac{\int_{B_2} S_{r_2}(f)\, df \int_{B_1} S_1(f)\, df}{\int_{B_2} S_{r_1}(f)\, df \int_{B_2} S_2(f)\, df} \tag{40}$$

where $S_{r_1}(f)$ is given by equation (21), $S_{r_2}(f)$ by equation (18), $S_1(f)$ by equation (29), and

$$\int_{B_2} S_2(f)\, df = R_2(0). \tag{41}$$

Making these substitutions and using the definition of α in equation (9) yields

$$\lim_{\tau_1 \to 0} \frac{S_2NR_0}{S_2NR_I} = \frac{2\alpha^2[E(1-\alpha) - (2\alpha - \alpha^2)K(1-\alpha)]}{\pi(1-\alpha)^2\left[\dfrac{\pi}{2} - \arcsin(1-\alpha) - (2\alpha - \alpha^2)^{\frac{1}{2}}\right]}. \tag{42}$$

This relative (interfering) signal-to-noise power ratio result is plotted in Fig. 3 and is particularly interesting since the plotted data can be viewed as a plot of S_2NR_0/S_2NR_I versus the input interfering signal-to-total broadband noise power ratio $S_2N_TR_I$. That is, it is seen that the ratio of average input interfering-signal power to total average input broadband noise power is given by

$$\mathrm{S_2N_TR_I} \triangleq \frac{R_2(0)}{R_1(0)} = \frac{1-\alpha}{\alpha}. \tag{43}$$

With this interpretation in mind, the plotted data show that there is a degradation in signal-to-noise power ratio in the signal band at all levels of input signal-to-noise power ratio as $\tau_1 \to 0$, and that this degradation increases monotonically with increasing input signal-to-total noise power ratio. We note the contrast of this result to that found by Davenport for the case in which the limiter is driven by an unmodulated sinusoid plus narrowband Gaussian noise where he shows that there is an enhancement in signal-to-noise ratio (measured in the narrow noise band) at high input signal-to-noise ratios.[5] It is also noted that the data plotted in Fig. 5 together with that in Fig. 3 show, that althougb the degradation increases monotonically with $\mathrm{S_2N_TR_I}$, it does not increase as rapidly as $W\tau_1(\mathrm{S_2NR_I})$ itself is increasing.

IV. CONCLUSIONS

This paper has concentrated on analyzing the power spectral density at the output of an ideal limiter when the input is driven by a narrowband gaussian signal plus an additive gaussian noise that consists of a broadband background component plus a narrowband interference. Conclusions that can be drawn from this work depend upon the system in which the limiter is used, and one is led to the following conclusions when this system consists of a spectrum analyzer preceded by the ideal limiter: Spectrum analyzer performance will be degraded by the presence of the limiter, and this degradation can be substantial when there is a strong narrowband interfering signal present at the limiter input. This intuitive conclusion follows from the fact that the signal-to-noise power ratio SNR measured in the signal band may be significantly degraded by the presence of the limiter when there is a strong narrowband interfering signal present at the limiter input, plus the fact that intermodulation products of the narrowband signal with the narrowband interference may be troublesome as indicated by a decreased signal-to-image power ratio SIR.

However, it is important to note that the results also indicate that the degradation in performance can be minimized by making the bandwidth observed by the limiter sufficiently wide that the average broadband noise power dominates both the signal and interference powers. This conclusion follows from the fact that such a procedure minimizes both the degradation in SNR and the decrease in SIR mentioned above since it ultimately requires that α approach unity. In particular, the

data plotted in Fig. 3 show that the signal-to-noise power ratio SNR is degraded by less than about 1.3 dB as long as the total average broadband noise power is greater than the average narrowband interference power. In addition, the data plotted in Fig. 4 show that the signal-to-image power ratio SIR is greater than about 14.5 dB as long as the total average broadband noise power is greater than the average narrowband interference power. This SIR result is interesting since it is indicative of the fact that intermodulation products do not grow as rapidly with increasing interfering-signal power in the situation analyzed here as they do when the ideal limiter is driven by two sinusoids plus narrowband Gaussian noise. This conclusion follows from comparison of Fig. 4 with the results of Jones as presented in his Fig. 4.[7] The difference in behavior appears to be due primarily to the fact that the strong narrowband signal in this analysis is a gaussian process and not a sinusoid.

It is of course true that the conclusions reached above based on the data plotted in Fig. 3 are conclusions based on the assumption that the broadband covariance function $R_1(\tau)$ is the triangular function specified in equation (20). This example was chosen as a typical example that is computationally convenient for studying the degradation in signal-to-noise power ratio SNR as a function of interfering-signal strength. It is also of interest to study the dependence of the degradation in SNR on the choice of $R_1(\tau)$, and it is noted that this can be accomplished by using $S_{Y_1}(f)$ given by equation (14) instead of $S_{Y_{1\Delta}}(f)$ given by equation (21) in the calculation of $\mathrm{SNR_0/SNR_I}$.

Finally, it is emphasized that the results leading to the above conclusions are asymptotic results that apply when the broadband noise correlation time τ_1 approaches zero. As discussed in Section I, our interest in small τ_1 stems from a desire to model the situation in which the average noise power in the spectral band occupied by the narrowband signal may be comparable to the average signal power but in which the total average noise power is much larger than the average signal power by virtue of the large noise bandwidth observed by the limiter. Thus we have a practical interest in the situation of small τ_1 , although it is of course true that the situation of engineering importance is that in which τ_1 although small is greater than zero; for example, $\alpha < 1$ makes physical sense only if $\tau_1 > 0$. With this in mind, it is of interest to determine the conditions that must be satisfied for the results of this work to be useful when $\tau_1 > 0$, and inspection of the analysis performed leads to the following conclusions (when the broadband noise covariance function $R_1(\tau)$ is written such that the band-

width of the broadband noise is approximately τ_1^{-1}): In order for the power spectral density result given by equation (11) and the signal-to-image power ratio result plotted in Fig. 4 to remain useful, it is necessary that certain conditions be satisfied:

(*i*) The broadband noise correlation time must itself satisfy the condition $\tau_1 \ll 1$.

(*ii*) The input signal-to-noise power ratio

$$\eta_S \triangleq \frac{R_S(0)}{R_N(0)} = \alpha \frac{R_S(0)}{C_0} \tau_1 \tag{10}$$

must satisfy the condition $\eta_S \ll 1$.

In addition to these conditions, in order for tbe power spectral density results given by (13) and (17) and the signal-to-noise power ratio results plotted in Fig. 3 and 5 to remain useful, it is necessary that the condition

$$\omega_i \tau_1 \ll 1, \qquad i = 0, 1, 2, \tag{44}$$

be satisfied. This last condition requires that the bandwidth of the broadband background noise be much larger tban tbe largest of the center frequencies ω_0, ω_1, and ω_2. The necessity of this condition was noted in Section I, and it was pointed out that this condition is not satisfied in communications situations in which the bandwidth of the broadband noise is much larger than that of the narrowband signals that may be present but much smaller than their center frequencies. However, inspection of the derivation of equations (13) and (17) shows that, if we set

$$\omega_0 = \omega_1 = \hat{\omega}_0/\tau_1 \qquad \text{and} \qquad \omega_2 = \hat{\omega}_0/\tau_1 + \omega_\Delta , \tag{45}$$

then we bave constructed a model for these "narrowband" communications situations for which equation (13) and (17) hold except for the term $S_{Y_1}(f)$ which is now given by

$$S_{Y_1}(f) = \frac{4}{\pi} \int_0^\infty \{\arcsin [\alpha \rho_1(\tau) + (1 - \alpha)\rho_2(\tau) \cos \omega_2 \tau] - \arcsin [(1 - \alpha)\rho_2(\tau) \cos \omega_2 \tau]\} \cos \omega \tau \, d\tau. \tag{46}$$

Signal-to-noise power ratio results corresponding to those plotted in Figs. 3 and 5 can be calculated (numerically) using equation (17) with $S_{Y_1}(f)$ given by equation (46) after making the simplifications that follow from the definitions of ω_0, ω_1, and ω_2 given in equation (45). When $\alpha = 1$ and $\hat{\omega}_0$ is large, the signal-to-noise ratio result corresponding to Fig. 3 will reduce to the result derived by Manasse, and others.[10]

APPENDIX A

Calculation of Output Power Spectral Density

Using the characteristic function method discussed hy Rice [Ref. 17] it can he shown [Ref. 12, p. 308] that, if the input to the ideal limiter of Fig. 1 is given by equation (1), then the autocorrelation function at the limiter output

$$R_Y(\tau) \triangleq \langle Y(t)Y^*(t-\tau)\rangle_{\text{av}} \tag{47}$$

is given by equation (7). Defining the input signal-to-noise power ratio η_S according to equation (10), it follows that

$$\begin{aligned} R_Y(\tau) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{2^{k+m}\Gamma^2[(k+m)/2]}{\pi^2 k!\, m!}\eta_S^k \\ &\qquad \cdot \left[\frac{\rho_S(\tau)}{1+\eta_S}\right]^k\left[\frac{\rho_N(\tau)}{1+\eta_S}\right]^m, \qquad k+m \quad \text{odd} \\ &= 0, \qquad \text{otherwise.} \end{aligned} \tag{48}$$

It was pointed out in the text that we are interested in the situation where η_S is small due to the large bandwidth of the broadband background noise. Motivated by this, it is noted that, upon summing on m, equation (48) can be written as

$$\begin{aligned} R_Y(\tau) &= \sum_{k=0(\text{even})}^{\infty} \frac{2^{k+1}}{\pi^2 k!}\Gamma^2\left(\frac{k+1}{2}\right) \\ &\quad \cdot {}_2F_1\left\{\frac{k+1}{2}, \frac{k+1}{2}; \frac{3}{2}; \left[\frac{\rho_N(\tau)}{1+\eta_S}\right]^2\right\}\frac{\rho_N(\tau)}{1+\eta_S}\left[\frac{\rho_S(\tau)}{1+\eta_S}\right]^k \eta_S^k \\ &\quad + \sum_{k=1(\text{odd})}^{\infty}\frac{2^k}{\pi^2 k!}\Gamma^2\left(\frac{k}{2}\right){}_2F_1\left\{\frac{k}{2}, \frac{k}{2}; \frac{1}{2}; \left[\frac{\rho_N(\tau)}{1+\eta_S}\right]^2\right\}\left[\frac{\rho_S(\tau)}{1+\eta_S}\right]^k \eta_S^k . \end{aligned} \tag{49}$$

Noting that ${}_2F_1(a, b; c; x)$ is finite for all $|x| < 1$ as long as $c \neq m \cdot (m = 0, -1, -2, \cdots)$† [Ref. 13, p. 556], it follows that

† Gauss's hypergeometric function is also absolutely convergent at $|x| = 1$ as long as Re $(c - a - b) > 0$. Thus in fact

$${}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; 1) = \frac{\pi}{2}$$

which implies that the series

$$\arcsin x = x + \frac{1}{2.3}x^3 + \frac{1.3}{2.4.5}x^5 + \cdots$$

converges for all $|x| \leq 1$.

$$R_Y(\tau) = \frac{2}{\pi}\left\{{}_2F_1\left[\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \left(\frac{\rho_N(\tau)}{1+\eta_S}\right)^2\right]\frac{\rho_N(\tau)}{1+\eta_S}\right.$$
$$\left. + {}_2F_1\left[\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \left(\frac{\rho_N(\tau)}{1+\eta_S}\right)^2\right]\frac{\rho_S(\tau)}{1+\eta_S}\eta_S\right\}$$
$$+ O[\eta_S^2\rho_S^2(\tau)\rho_N(\tau)] + O[\eta_S^3\rho_S^3(\tau)] \qquad (50)$$

as $\eta_S \to 0$, for all τ such that $|\rho_N(\tau)| < 1$. Moreover, by expressing the hypergeometric function in the first term of equation (50) in its series form and then appropriately collecting terms, it can be shown that

$$\frac{\rho_N(\tau)}{1+\eta_S}\,{}_2F_1\left[\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \left(\frac{\rho_N(\tau)}{1+\eta_S}\right)^2\right]$$
$$= \arcsin \rho_N(\tau) - \rho_N(\tau)[1 - \rho_N^2(\tau)]^{-\frac{1}{2}}\eta_S + O[\eta_S^2\rho_N(\tau)] \qquad (51)$$

as $\eta_S \to 0$, for all τ such that $|\rho_N(\tau)| < 1$. Also, it is immediately recognized that, in the second term in equation (50),

$$\frac{\rho_S(\tau)}{1+\eta_S}\,{}_2F_1\left[\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \left(\frac{\rho_N(\tau)}{1+\eta_S}\right)^2\right]$$
$$= \rho_S(\tau)[1 - \rho_N^2(\tau)]^{-\frac{1}{2}} + O[\eta_S\rho_S(\tau)] \qquad (52)$$

as $\eta_S \to 0$, for all τ such that $|\rho_N(\tau)| < 1$. Therefore, recalling that the noise $N(t)$ contains a broadband component so that in fact

$$|\rho_N(\tau)| < 1 \qquad (53)$$

for all $|\tau| > 0$,† it is concluded upon substitution of equations (51) and (52) into equation (50) that

$$R_Y(\tau) = \frac{2}{\pi}\{\arcsin \rho_N(\tau) + [\rho_S(\tau) - \rho_N(\tau)][1 - \rho_N^2(\tau)]^{-\frac{1}{2}}\eta_S\}$$
$$+ O[\eta_S^2\rho_S(\tau)] + O[\eta_S^2\rho_N(\tau)] \qquad (54)$$

as $\eta_S \to 0$, for all $|\tau| > 0$.

In order to calculate the power spectral density $S_Y(f)$ at the limiter output it is necessary to evaluate

$$S_Y(f) \triangleq 2\int_0^\infty R_Y(\tau)\cos\omega\tau\, d\tau, \qquad \omega \triangleq 2\pi f. \qquad (55)$$

† Note that this follows from the integrability condition placed on the broadband covariance function $R_1(\tau)$ by (5). This integrability condition implies that $|\rho(x)| < \rho(0)$ for all $|x| > 0$ and requires that the power spectrum of the broadband noise contain no line components.

As it stands, $R_Y(\tau)$ given by equation (54) is not enough because of the difficulty as $\tau \to 0$. It is not clear from the foregoing analysis whether or not the representation given by equation (54) is valid as $\tau \to 0$ when $\eta_S \to 0$, and in fact this representation may be valid for all $\rho_S(\tau)$ and $\rho_N(\tau)$ of interest [compare Ref. 18].* In any event, the difficulties involved in evaluating the remainder terms in order to examine this possibility can be circumvented by using the well-known result that $R_Y(\tau)$ is also given by equation (8).[3] Thus,

$$R_Y(\tau) = \frac{2}{\pi} \arcsin \frac{\rho_N(\tau) + \eta_S \rho_S(\tau)}{1 + \eta_S} \tag{56}$$

which implies that

$$R_Y(\tau) = \frac{2}{\pi} \arcsin \rho_N(\tau) + o(1) \tag{57}$$

as $\eta_S \to 0$, uniformly in τ. In fact, making use of the expressions for $R_Y(\tau)$ given by equations (54) and (57) in conjunction with the expression for η_S given by equation (10) and the integrability condition in equation (5), it is seen that, if $R_1(\tau)$ can be written in the form specified by equation (3) and the parameters α and $R_S(0)/C_0$ satisfy the conditions $\alpha > 0$, $R_S(0)/C_0 < \infty$, then $R_Y(\tau)$ can be expressed:†

$$\begin{aligned} R_Y(\tau) &= \frac{2}{\pi} \arcsin \rho_N(\tau) + o(1), \qquad 0 \leqq |\tau| \leqq \tau_1 \\ &= \frac{2}{\pi} \left\{ \arcsin \rho_N(\tau) + \alpha \frac{R_S(0)}{C_0} [\rho_S(\tau) - \rho_N(\tau)][1 - \rho_N^2(\tau)]^{-\frac{1}{2}} \tau_1 \right\} \\ &\quad + O[\tau_1^2 \rho_S(\tau)] + O[\tau_1^2 \rho_N(\tau)], \qquad |\tau| \geqq \tau_1 , \end{aligned} \tag{58}$$

sa $\tau_1 \to 0$. Substituting this result into equation (55) and assuming that the integrability conditions

$$\int_0^\infty |\rho_S(\tau)| \, d\tau < \infty \tag{59}$$

$$\int_0^\infty |\rho_N(\tau)| \, d\tau < \infty \tag{60}$$

are satisfied, there results

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* McFadden derives a similar expression for the case of a weak sinusoid in additive gaussian noise and asserts that the expansion is valid at $\tau = 0$ as long as $\rho_N(\tau)$ satisfies certain differentiability conditions.

† Another method for obtaining equation (58) is to expand equation (56) in a Taylor series about $\rho_N(\tau)$.

$$S_Y(f) = \frac{4}{\pi}\left\{\int_0^\infty \arcsin \rho_N(\tau) \cos \omega\tau \, d\tau \right.$$
$$\left. + \alpha \frac{R_S(0)}{C_0} \tau_1 \int_{\tau_1}^\infty [\rho_S(\tau) - \rho_N(\tau)][1 - \rho_N^2(\tau)]^{-\frac{1}{2}} \cos \omega\tau \, d\tau \right\} + o(\tau_1) \tag{61}$$

as $\tau_1 \to 0$, uniformly iu f. This result can immediately be simplified by observing that the predominant contributions to $S_Y(f)$ due to interaction of the signal and noise processes are due to interaction of the signal process with the narrowband interference component of the noise. In fact, noting that

$$\rho_N(\tau) = \alpha\rho_1(\tau) + (1 - \alpha)\rho_2(\tau) \cos \omega_2\tau, \tag{62}$$

it can be seen that equation (61) reduces to equation (11).

APPENDIX B

Derivation of Output-Power Spectral Density Expansion

It is shown in Appendix A that the output power spectral density can be expressed according to equation (11); namely, that

$$S_Y(f) = S_{Y_N}(f) + \frac{4}{\pi} \alpha \frac{R_S(0)}{C_0} \tau_1 \int_0^\infty [\rho_S(\tau) - (1 - \alpha)\rho_2(\tau) \cos \omega_2\tau]$$
$$\cdot [1 - (1 - \alpha)^2 \rho_2^2(\tau) \cos^2 \omega_2\tau]^{-\frac{1}{2}} \cos \omega\tau \, d\tau + o(\tau_1) \tag{63}$$

as $\tau_1 \to 0$, uniformly in f, where

$$S_{Y_N}(f) \triangleq \frac{4}{\pi} \int_0^\infty \arcsin [\alpha\rho_1(\tau) + (1 - \alpha)\rho_2(\tau) \cos \omega_2\tau] \cos \omega\tau \, d\tau \tag{64}$$

is the output power spectral density when the noise $N(t)$ alone is present at the limiter input. $S_Y(f)$ can be put in a more useful form by expanding both $[1 - (1 - \alpha)^2\rho_2^2(\tau) \cos^2 \omega_2\tau]^{-\frac{1}{2}}$ and $\arcsin [\alpha\rho_1(\tau) + (1 - \alpha)\rho_2(\tau) \cos \omega_2\tau]$. Proceeding with expansion of the latter it is seen that [Ref. 13, item 15.1.6]

$$\arcsin [\alpha\rho_1(\tau) + (1 - \alpha)\rho_2(\tau) \cos \omega_2\tau]$$
$$= \frac{1}{2\pi^{\frac{1}{2}}} \sum_{m=0}^\infty \frac{\Gamma^2(m + \frac{1}{2})}{\Gamma(m + \frac{3}{2})m!} [\alpha\rho_1(\tau) + (1 - \alpha)\rho_2(\tau) \cos \omega_2\tau]^{2m+1}$$
$$= \frac{1}{2\pi^{\frac{1}{2}}} \sum_{m=0}^\infty \frac{\Gamma^2(m + \frac{1}{2})}{\Gamma(m + \frac{3}{2})m!} \sum_{j=0}^{2m+1} \frac{(2m + 1)!}{(2m + 1 - j)!\, j!}$$
$$\cdot [(1 - \alpha)\rho_2(\tau) \cos \omega_2\tau]^j [\alpha\rho_1(\tau)]^{2m+1-j}$$

$$= \arcsin \left[(1-\alpha)\rho_2(\tau) \cos \omega_2\tau\right] + \frac{1}{2\pi^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{\Gamma^2(m+\frac{1}{2})}{\Gamma(m+\frac{3}{2})m!}$$

$$\cdot \sum_{j=0}^{2m} \frac{(2m+1)!}{(2m+1-j)!\,j!} \left[(1-\alpha)\rho_2(\tau) \cos \omega_2\tau\right]^j \left[\alpha\rho_1(\tau)\right]^{2m+1-j}. \tag{65}$$

Thus, substituting equation (65) into equation (64), we have

$$S_{Y_N}(f) = S_{Y_1}(f) + S_{Y_2}(f), \tag{66}$$

where

$$S_{Y_1}(f) = \frac{2}{\pi^{\frac{3}{2}}} \int_0^\infty \sum_{m=0}^{\infty} \frac{\Gamma^2(m+\frac{1}{2})}{\Gamma(m+\frac{3}{2})m!} \sum_{j=0}^{2m} \frac{(2m+1)!}{(2m+1-j)!\,j!}$$

$$\cdot \left[(1-\alpha)\rho_2(\tau) \cos \omega_2\tau\right]^j \left[\alpha\rho_1(\tau)\right]^{2m+1-j} \cos \omega\tau \, d\tau \tag{67}$$

and

$$S_{Y_2}(f) = \frac{4}{\pi} \int_0^\infty \arcsin \left[(1-\alpha)\rho_2(\tau) \cos \omega_2\tau\right] \cos \omega\tau \, d\tau. \tag{68}$$

We have succeeded in breaking $S_Y(f)$ into a broadband component $S_{Y_1}(f)$ plus a component $S_{Y_2}(f)$ consisting of narrowband contributions. In fact, letting $x \triangleq \tau/\tau_1$, it can be seen, using the integrability condition of equation (5), that

$$\int_0^\infty \left[(1-\alpha)\rho_2(\tau) \cos \omega_2\tau\right]^j \left[\alpha\rho_1(\tau)\right]^{2m+1-j} \cos \omega\tau \, d\tau$$

$$= \tau_1 \int_0^\infty \left[(1-\alpha)\rho_2(\tau_1 x) \cos \omega_2\tau_1 x\right]^j \left[\alpha\rho(x) \cos \omega_1\tau_1 x\right]^{2m+1-j} \cos \omega\tau_1 x \, dx$$

$$= \tau_1 \int_0^\infty (1-\alpha)^j \alpha^{2m+1-j} \rho^{2m+1-j}(x) \, dx + o(\tau_1) \tag{69}$$

as $\tau_1 \to 0$, for all $f \leqq f_{\max} < \infty$ for arbitrary fixed $f_{\max}$, as long as $j < 2m + 1$. Moreover, using this integrability condition plus the fact that the series in the integrand is absolutely convergent, it can be shown that

$$S_{Y_1}(f) = \frac{2}{\pi^{\frac{3}{2}}} \tau_1 \int_0^\infty \sum_{m=0}^{\infty} \frac{\Gamma^2(m+\frac{1}{2})}{\Gamma(m+\frac{3}{2})m!} \sum_{j=0}^{2m} \frac{(2m+1)!}{(2m+1-j)!\,j!}$$

$$\cdot (1-\alpha)^j \left[\alpha\rho(x)\right]^{2m+1-j} \, dx + o(\tau_1) \tag{70}$$

as $\tau_1 \to 0$, for all $f \leqq f_{\max} < \infty$, which can be written as

$$S_{Y_1}(f) = \frac{4}{\pi} \tau_1 \int_0^\infty \{\arcsin \left[\alpha\rho(x) + 1 - \alpha\right]$$

$$- \arcsin (1-\alpha)\} \, dx + o(\tau_1) \tag{71}$$

as $\tau_1 \to 0$, for all $f \leqq f_{\max} < \infty$ for arbitrary fixed $f_{\max}$. Thus it is seen that the broadband component $S_{Y_1}(f)$ becomes white across any frequency band of finite extent as $\tau_1 \to 0$ and moreover that, if $\alpha = 1$, then $S_{Y_1}(f)$ is just the output power spectral density that would be observed if the broadband component of the noise was present alone at the limiter input.

Turning now to $S_{Y_2}(f)$ given by equation (68), it is seen that

$$\begin{aligned}
&\arcsin\,[(1-\alpha)\rho_2(\tau)\,\cos\omega_2\tau] \\
&\qquad = \frac{1}{2\pi^{\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{\Gamma^2(k+\frac{1}{2})}{\Gamma(k+\frac{3}{2})k!}\,[(1-\alpha)\rho_2(\tau)\,\cos\omega_2\tau]^{2k+1} \\
&\qquad = \frac{1}{2\pi^{\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{\Gamma^2(k+\frac{1}{2})}{\Gamma(k+\frac{3}{2})k!}\,[(1-\alpha)\rho_2(\tau)]^{2k+1} \\
&\qquad\qquad \cdot\sum_{r=0}^{k}\frac{(2k+1)!}{(2k+1-r)!\,r!\,2^{2k}}\cos\,(2k+1-2r)\omega_2\tau.
\end{aligned} \tag{72}$$

Now, letting $k - r \triangleq m$ and then interchanging the order of summation on k and m, there results

$$\begin{aligned}
&\arcsin\,[(1-\alpha)\rho_2(\tau)\,\cos\omega_2\tau] \\
&\quad = \frac{1}{2\pi^{\frac{1}{2}}}\sum_{m=0}^{\infty}\sum_{k=m}^{\infty}\frac{\Gamma^2(k+\frac{1}{2})(2k+1)!}{\Gamma(k+\frac{3}{2})k!\,(k+m+1)!\,(k-m)!\,2^{2k}} \\
&\quad\quad \cdot[(1-\alpha)\rho_2(\tau)]^{2k+1}\cos\,(2m+1)\omega_2\tau.
\end{aligned} \tag{73}$$

However [Ref. 13, item 6.1.18],

$$(2k+1)! = (2\pi)^{-\frac{1}{2}}2^{2k+\frac{3}{2}}\Gamma(k+1)\Gamma(k+\tfrac{3}{2}) \tag{74}$$

so that equation (73) can be rewritten as

$$\begin{aligned}
&\arcsin\,[(1-\alpha)\rho_2(\tau)\,\cos\omega_2\tau] \\
&= \frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{k=m}^{\infty}\frac{\Gamma^2(k+\frac{1}{2})}{(k+m+1)!\,(k-m)!}\,[(1-\alpha)\rho_2(\tau)]^{2k+1}\cos\,(2m+1)\omega_2\tau \\
&= \frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma^2(j+m+\frac{1}{2})}{\Gamma(j+2m+2)j!}\,[(1-\alpha)\rho_2(\tau)]^{2j+2m+1}\cos\,(2m+1)\omega_2\tau \\
&= \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{\Gamma^2(m+\frac{1}{2})}{\Gamma(2m+2)}\,{}_2F_1\{m+\tfrac{1}{2},\,m+\tfrac{1}{2};\,2m+2;\,[(1-\alpha)\rho_2(\tau)]^2\} \\
&\quad\cdot[(1-\alpha)\rho_2(\tau)]^{2m+1}\cos\,(2m+1)\omega_2\tau.
\end{aligned} \tag{75}$$

Substituting this result into equation (68), we obtain the result stated in equation (15).

The expansion of $[1 - (1 - \alpha)^2 \rho_2^2(\tau) \cos^2 {}_2\tau]^{-\frac{1}{2}}$ in the second term in equation (63) can be pursued in a manner identical to that used above for the expansion of arcsin $[(1 - \alpha)\rho_2(\tau) \cos \omega_2 \tau]$, and the result obtained is that given in (13).

It is pointed out in the text that the assumption $\rho_2(\tau) \equiv 1$ greatly simplifies the expression for $S_Y(f)$ without obscuring the most important effects that result from the presence of narrowband interference. In particular, it is seen that the assumption $\rho_2(\tau) \equiv 1$ violates the integrability condition in equation (60). As a result, equation (13) does not hold uniformly in f under this assumption since the points $f = \pm k f_2$, $k = 1, 3, \cdots$, must be excluded. However, it is observed tbat equation (13) can be made to hold at these points as $\tau_1 \to 0$ by addition of the remainder term

$$O\left(\tau_1^2 \int_0^\infty \mid \rho_2(\tau) \mid d\tau\right). \tag{76}$$

Moreover, it is seen from equation (15) that, when $\rho_2(\tau) \equiv 1$, $S_{Y_2}(f)$ is nonzero only at $f = \pm k f_2$, $k = 1, 3, \cdots$, and its value at these points is

$$O\left(\int_0^\infty \mid \rho_2(\tau) \mid d\tau\right). \tag{77}$$

Thus in fact it can be seen that, when $\rho_2(\tau) \equiv 1$, it is meaningful to write $S_Y(f)$ as given by equation (17).

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